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Patterns in dissipative systems with weakly broken continuous symmetry

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Patterns in dissipative systems with weakly broken symmetry are studied based upon the simplest canonical equation (generalized Nikolaevskiy model). A generic cubic dispersion equation governing stability of steady spatially periodic patterns is derived and analyzed. A domain of stable states in the space of the problem parameters (stability balloon) is obtained. It is shown that the domain is characterized by unusual scaling properties, so that its different parts obey different scalings. The results obtained may be applied to describe instabilities of advancing fronts and interfaces, pattern formation in reaction-diffusion systems, nonlinear evolution of seismic waves, and other phenomena.

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The theory of turbulence is one of the most appealing topics of the entire area of hydrodynamics and statistical physics. Owing to the great complexity of the phenomenon a general theory does not exist and is unlikely ever to be created. In this situation any tractable model shedding light on certain aspects of the problem is of great importance. In the present paper such a model is proposed and analyzed. The model describes so-called soft-mode turbulence (SMT) in the case of weakly broken symmetry.

SMT, also known as the Nikolaevskiy chaos, is an unusual type of chaos that arises directly at onset in dissipative systems with a short-wavelength instability and a continuous family of unstable spatially uniform states [1–18] . All the states are physically equivalent and reduced to each other by transformations of a certain group of symmetry. It should be emphasized that the only important point is the continuity of the symmetry group, while the type of the corresponding symmetry transformations does not play a substantial role. The latter makes the phenomenon quite general (see below).

On account of the symmetry the stability spectrum of the system has a branch of modes whose decay rate vanishes in the long-wavelength limit (Goldstone branch). SMT is associated with the interplay of long-wavelength modes from this branch with those related to the short-wavelength instability. The turbulence arises directly from the spatially uniform states as a result of a single supercritical bifurcation. On the one hand, the phenomenon may be regarded as an analog of the second-order phase transitions (with the critical slowing-down, divergence of the correlation length at the transition point, etc.) where average amplitudes of the turbulent modes play the role of the order parameter. On the other hand it exhibits the continuous spectrum and the Kolmogorov cascades (both normal and inverse) typical to hydrodynamic turbulence. For more details see, e.g., Ref. [1].

These and other peculiarities of SMT motivated its extensive study both experimentally (e.g., Refs. [2–5]) and theoretically [1,6,7,9–14]. The latter mostly focused either on phenomenological amplitude equations introduced to describe SMT at electroconvection [8,14] or on the Nikolaevskiy equation [1,6,7,10–13].

Initially proposed to describe seismic waves in the Earth's crust [15] the Nikolaevskiy equation arises in a variety of other problems, such as pattern formation in reaction-diffusion systems [16,17], certain transversal instabilities of traveling fronts of chemical reactions and phase transitions [1,18] including the laser ablation [19], and even in acoustic stimulation of oil wells [20]. The equation may be regarded as the simplest canonical model to study SMT and related phenomena beyond the framework of amplitude equations. It should be stressed that owing to the interplay of different scales typical for SMT [1], the transition from "microscopic" to amplitude equations is not a straightforward matter in this case [1,21].

Note, however, that in any real situation the symmetry can never be perfect—it is always broken at least weakly due to effects of lateral boundary conditions, spatial inhomogeneities, external field(s), etc. The symmetry violation should result in damping of the Goldstone mode, which in turn may change the dynamical properties of the system qualitatively. The issue is of great importance—variations in the range of the symmetry violation caused by an external field provide the only opportunity for observing a smooth crossover from the Turing-type patterns (at strong symmetry breaking) to SMT (at the unbroken symmetry) exhibited by one and the same physical system [4,5]. Meanwhile, theoretical analysis of these effects performed in Refs. [8,14] is based on amplitude equations which do not describe one of the basic attributes of SMT — the interplay of different scales. Then the question arises: Which of the features discussed in Refs. [8,14] are generic to SMT and which are specific to the models analyzed?

The present paper is an attempt to answer this question based upon a systematic study of a generalized version of the one-dimensional Nikolaevskiy model. Two-dimensional versions of the problem are not considered, though it may be expected that the obtained results are valid for two-dimensional cases at least qualitatively. The analysis is based upon a perturbation theory being applicable at small ϵ and μ, where ϵ is the usual control parameter and μ stands for the associated with the broken symmetry damping rate of the

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Goldstone mode. A single generic dispersion equation determining the stability conditions for steady spatially periodic patterns (SSPP) regardless of the scaling properties of the equation coefficients is derived and inspected. A stability (Busse) balloon in the form $k_{\mathrm{Left}}(\mu,\epsilon)\le k\le k_{\mathrm{Right}}(\mu,\epsilon)$ is obtained, where $k=2\pi/\lambda$ and λ stands for the period of stable SSPP. The balloon exists only at $\epsilon\le\epsilon_c(\mu)$ and vanishes at $\epsilon=\epsilon_c(\mu)$ while the symmetry still remains broken.

The canonical form of the Nikolaevskiy equation is as follows [15]:

$$\frac{\partial v}{\partial t}+\frac{\partial^2}{\partial x^2}\left[\epsilon-\left(1+\frac{\partial^2}{\partial x^2}\right)^2\right]v+v\frac{\partial v}{\partial x}=0, \quad (1)$$

with purely real v and ϵ. Usually the Goldstone modes in the stability spectra of both trivial ($v=0$) and steady nontrivial solutions of Eq. (1) are linked to its Galilean invariance [12]. This is not quite correct. The Galilean transformation ($x\to x+v_0t$; $v\to v+v_0$) makes a steady solution time-dependent and therefore cannot generate a neutrally stable, time-independent mode. The actual reason for the Goldstone branches associated with Eq. (1) is equivalence of this equation to one with the same linear part and nonlinearity v_x^2 [6,22], which is invariant to transformation $v\to v+\text{const}$.

The stability analysis of the trivial solution $v=0$ to perturbations $\delta v\sim\exp(\gamma t+ikx)$ brings about the spectrum

$$\gamma_k=k^2[\epsilon-(k^2-1)^2]. \quad (2)$$

Here, and in what follows, the gamma subscript indicates the corresponding value of the wave number. In agreement with the problem symmetry, $\gamma_0=0$. Broken symmetry should result in $\gamma_0=-\mu<0$. In this case, the dispersion relation reads

$$\gamma_k=-\mu+k^2[\mu+\epsilon-(k^2-1)^2], \quad (3)$$

where μ in square brackets is added to keep the same meaning for ϵ as that in Eq. (2). In the case of symmetry violation by an external field which does not break left-right parity, μ is proportional to the square of the field, see Refs. [4,5,8,14]. Eq. (3) gives rise to the generalized Nikolaevskiy equation of the form

$$\frac{\partial v}{\partial t}+\left\{\mu+\frac{\partial^2}{\partial x^2}\left[\mu+\epsilon-\left(1+\frac{\partial^2}{\partial x^2}\right)^2\right]\right\}v+v\frac{\partial v}{\partial x}=0. \quad (4)$$

Let us designate the local maximum of γ_k achieved at $k=k_{\max}=O(1)$, see Eq. (3), as $\gamma_{\max}$. At any $\gamma_{\max}>0$ Eq. (4) has a continuous family of spatially periodic solutions [23]

$$v(x,t)=\sum_{n=-\infty}^{\infty}V_{nk}(t)\exp(inkx),\quad V_{nk}^*=V_{-nk}. \quad (5)$$

Here the wave number k may take any value from the range $\gamma_k>0$. Inserting Eq. (5) into Eq. (4), one obtains a chain of coupled equations

$$\frac{dV_{nk}}{dt}=\gamma_{nk}V_{nk}+ik\sum_m mV_{mk}V_{(n-m)k} \quad (6)$$

and a detached equation for V_0

$$\frac{dV_0}{dt}=-\mu V_0. \quad (7)$$

Thus, for steady solutions of Eqs. (6) and (7), V_0 vanishes. Regarding other amplitudes, employing the fact that $V_{(n+1)k}=o(V_{nk})$ for small $\gamma_{\max}$ and solving the steady version of Eq. (6) by iterations, it is easy to obtain

$$|V_k|^2=-\frac{\gamma_k\gamma_{2k}}{k^2}[1+O(\gamma_{\max})],$$

$$V_{2k}=-\frac{ikV_k^2}{\gamma_{2k}}[1+O(\gamma_{\max})],\ \dots\ . \quad (8)$$

Note that in Eq. (8) $\gamma_k>0$, while $\gamma_{2k}<0$, so $|V_k|^2$ is a positive quantity, as it should be.

The key point of the present study is the stability analysis of the SSPP. To this end, let us consider small perturbations to the steady solutions in the form

$$\delta v=\sum_n U_{nk+p}\exp[\sigma t+i(nk+p)x], \quad (9)$$

where $k=2\pi/\lambda$ parameterizes the nonlinear solution, whose stability is analyzed, and small p stands for the perturbation wave number. For the given perturbations linearization of Eq. (4) about the steady solution Eq. (8) gives rise to the following equations:

$$(\sigma-\gamma_{nk+p})U_{nk+p}+i(nk+p)\sum_{m=-\infty}^{\infty}V_{n-m}U_{mk+p}=0. \quad (10)$$

The solvability condition, which in this case is reduced to equalization of the determinant of Eq. (10) to zero, results in an infinite number of branches $\sigma_n(k,p)$ in the stability spectrum. However, at small ε, μ and p, all of them except three (at $n=0,\pm1$) are stable and have the form $\sigma_n=\gamma_{nk+p}+o(1)$; $n\neq0,\pm1$. For the remaining three branches, γ_{nk+p} is itself a small quantity. At small γ_{nk+p}, the approximation $\sigma_n\approx\gamma_{nk+p}$ does not hold, and these three branches should be inspected more carefully. Employing the smallness of ϵ for the evaluation of the determinant by a perturbation theory, after certain algebra the following cubic dispersion equation, describing the three "dangerous" branches, may be obtained:

$$\sigma^3+a_1\sigma^2+a_2\sigma+a_3=0, \quad (11)$$

where

$$a_1=2\gamma_k-\gamma_1''p^2-\gamma_p, \quad (12)$$

$$a_2=-[(2\gamma_2+\gamma_1'')\gamma_k+(\gamma_k')^2]p^2-[2\gamma_k-\gamma_1''p^2]\gamma_p+(\gamma_1''/2)^2p^4, \quad (13)$$

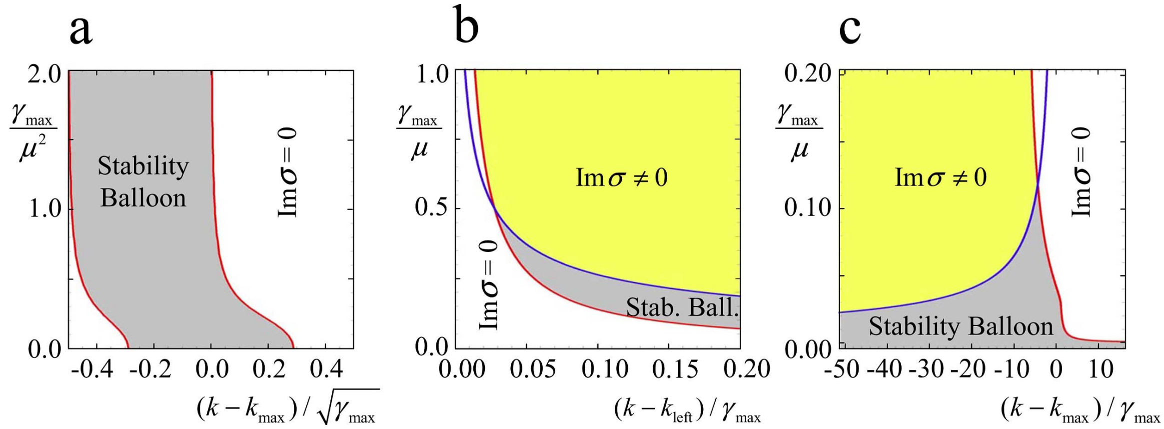

FIG. 1. (Color online) The base (a) and tips (b, c) of the stability balloon (gray) for steady specially periodic solutions of Eq. (4). Note the different scales of axis on panels (a–c). White and yellow domains correspond to monotonic and oscillatory instabilities, respectively.

$$a_3 = -\frac{2\gamma_k\gamma_k'\gamma_{2k}}{k}p^2 + 4(\gamma_2 - \gamma_2')\gamma_k^2 p^2 + [\gamma_1''\gamma_k + (\gamma_k')^2 - (\gamma_1''/2)^2 p^2]\gamma_p p^2 + \left(\gamma_1'' - \frac{\gamma_1'''}{3}\right)\gamma_2\gamma_k p^4. \tag{14}$$

Here prime denotes derivative with respect to k.

It should be emphasized that the smallness of ϵ is the only applicability condition for Eqs. (11)–(14)—the expressions remain the same at any value of μ. For this reason, and since Eqs. (11)–(14) do not employ an explicit form of dependence γ_k, they are generic and robust with respect to possible ways of generalizing of Eq. (1). The equations provide the opportunity to study the stability problem in different scaling ranges, as well as to describe crossovers between the ranges.

To answer the stability question, one must inspect the signs of the real parts of the three roots $\sigma_{0,\pm 1}$ of Eq. (11) as functions of μ, ϵ, k, and p. It should be stressed that for certain values of k, unstable modes are separated from sideband perturbations ($p \to 0$) by a finite gap. This makes the sideband limit insufficient for the stability analysis, so that the inspection involving finite p is required. Details of this inspection are rather cumbersome and will be discussed elsewhere. Here only the final results are presented.

It is shown that for $\mu \ll 1$, there are two characteristic scales of ϵ relative to μ, namely, $\epsilon = O(\mu^2)$ and $\epsilon = O(\mu)$. For $\epsilon = O(\mu^2)$ the stability condition reads as follows:

$$1 - 12K^2 - 72E^2K(1 - 4K^2) > 0. \tag{15}$$

Here $K = (k - k_{max})/\sqrt{\gamma_{max}}$, $k_{max} \simeq 1 + (\mu/4)$, $E = \gamma_{max}/\mu^2$ and $\gamma_{max} \simeq \epsilon + (\mu^2/4)$. Note in this case, that a single condition determines both boundaries of the stability balloon. Violation of Eq. (15) because of crossing of either the left or right boundaries of the stability balloon results in monotonic instability (Im$\sigma \equiv 0$).

If $|\epsilon| \ll \mu^2$ and $k \to 0$, the damping of the long-wavelength modes is large with respect to γ_{max}. In this case Eq. (15) is reduced to the usual Eckhaus condition [24] $-1/2\sqrt{3} \leq K \leq 1/2\sqrt{3}$. An increase in ϵ brings about a shift of the stability balloon to the left, but hardly affects its width, see Fig. 1(a). When $\mu^2 \ll \varepsilon \ll \mu$, the right boundary of the balloon tends to $K = 0$ (which corresponds to $\gamma_k = \gamma_{max}$), and the left to $K = -1/2$ (corresponding to the left root of equation $\gamma_k = 0$). Since destabilization of SSPP occurs owing to coupling of modes with k close to k_{max} with those from the vicinities of $k = 0$ and $k = 2k_{max}$, the observed asymmetry of the balloon is obviously related to the difference in dissipation rates for these two subbands, which becomes pronounced for $\mu^2 \ll \epsilon$.

A further increase in ϵ transfers us to the region $\epsilon = O(\mu)$, where $\gamma_{max} \simeq \epsilon$. This increase gives rise to splitting of the balloon into two narrow tongues [each with the characteristic width of $O(\gamma_{max})$] separated by a gap, see Figs. 1(b) and 1(c). One tongue is situated in the vicinity of $k = k_{max}$. It ends up with a cusp at $\epsilon = \nu_c\mu$ with $\nu_c \simeq 0.1184\ldots$ and $(k - k_{max})/\epsilon \simeq -4.299\ldots$.

The other tongue lies close to the left margin of the neutral stability boundary for the trivial state $v = 0$, i.e., to the left root k_{left} of equation $\gamma_k = 0$ [25]. The tongue is defined by the condition

$$\frac{\mu}{72\epsilon} \leq \frac{k - k_{left}}{\epsilon} \leq \frac{\mu^2}{144\epsilon^2}. \tag{16}$$

It vanishes with a cusp at $\epsilon = \mu/2$, see Fig. 1(b). Thus, at $\nu_c\mu \leq \epsilon \leq \mu/2$ this tongue represents the only domain of stable rolls. For $\epsilon > \mu/2$ the entire set of SSPS is unstable. Vanishing of the balloon at the finite value of μ is connected with a "resource of instability" of Eq. (1)—none of its steady spatially periodic solutions are even marginally stable to long-wavelength modulations [6]. In this case a certain finite stabilization of the long-wavelength modes is required to suppress the instability. Note also that in contrast to the case $\epsilon = O(\mu^2)$ now the left and right boundaries of each tongue are defined by two different, absolutely independent conditions. Accordingly, violation of one of the conditions (crossing the left boundary of the left tongue and the right boundary of the central one) triggers a monotonic instability, while violation of the other (crossing the opposite boundaries) initiates an oscillatory instability, see Figs. 1(b) and 1(c).

Independence of the two conditions means that for both the tongues, the cusp tips are just points of intersections of two smooth curves, while for each of the curves individually these points are not singled out in any way. Therefore, completion of the stability balloon with a cusp, should be a generic feature of the problem, see Refs. [8,14].

An intermediate scaling range corresponding to the crossover from $\epsilon = O(\mu^2)$ to $\epsilon = O(\mu)$ does not admit simple analytical study. However, it should be stressed that Eqs. (11)–(14) are valid in this range too and still may be employed to study (e.g., numerically) the smooth transformation of the balloon base [Fig. 1(a)] into its tips [Figs. 1(b) and 1(c)].

In conclusion, it may be said that a generic cubic dispersion equation governing stability of SSPS in the generalized Nikolaevskiy equation has been derived and analyzed. Comparison of the results obtained with those discussed in Refs. [8,14] shows that along with certain similarities (a stability balloon popping up at a broken symmetry, independent stability conditions for different boundaries of the balloon, etc.) there are striking differences generally related to the interplay of the various scales exhibited by the Nikolaevskiy model (different scaling for different parts of the balloon with the corresponding crossovers between them), which do not have the amplitude equations inspected in Ref. [8,14].

Cox and Matthews have recently published a paper where another version of the damped Nikolaevskiy equation was discussed within the formalism of amplitude equations [18]. In contrast to the generic Eqs. (11)–(14), the scale mixing did not allow them to derive a single set of amplitude equations valid for the entire balloon, so various equations valid for the specific scales each were introduced. Despite the difference in the versions of the damped Nikolaevskiy model, the results obtained in their study are identical to those discussed above, which proves the aforementioned robustness of Eqs. (11)–(14).

The changes of the scaling in different parts of the stability balloon and the various types of instabilities arising on crossing its boundaries provide grounds for a diversity of dynamical phase transitions which may be observed in this problem [18]. Systematic study of these transitions, and the corresponding patterns is a fascinating issue.

Experimental study of SMT for electroconvection in a homeotropically aligned nematic layer with the symmetry violated by an external magnetic field applied in the layer plane detected a stability balloon for SSPP [4,5]. However, scaling properties of the balloon have not been inspected in detail. The author believes that the present paper may motivate this study along with further theoretical analysis of SMT.

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